Problem Set 6

1. The Hadamard matrix H_n is a $2^n \times 2^n$ matrix which is defined by the recurrence $H_0 = [1]$ and

$$H_{n+1} = \begin{bmatrix} H_n & H_n \\ H_n & \mathbf{0}_n \end{bmatrix}$$

where $\mathbf{0}_n$ is a $2^n \times 2^n$ all zeros matrix.

   (a) What is the usual runtime of multiplying a vector by an $N \times N$ matrix?

   (b) Write down H_3.

   (c) Let $N = 2^n$. Give an $O(N \log N)$ algorithm to compute $H_n v$ where v is a given vector of length N. (This is sometimes used to produce fast sketches for numerical linear algebra.)

2. You're given a collection of lines $y = m_i x + b_i$ in the plane such that no three lines meet in a point. Say that line i is "visible from above" if there there is a vertical line for which line i intersects it at greater y-coordinate than any other line.

   Give an $O(n \log n)$ algorithm to find all the lines that are visible from above.

3. (a) Suppose that you have a black-box algorithm which for arrays of size at least 4 can produce can produce an element between the first and third quartile in $O(1)$ time. Using this black box, show how to find the median of an array in linear time. Assume the list has odd length for simplicity.

   (b) How could you approximate this black-box in practice?

4. You're given an $n \times n$ array containing a number x_{ij} in row i column j. We say that (i, j) is a local minimum of the array if x_{ij} is smaller than or equal to each of its horizontally and vertically adjacent neighbors.

Show how to find a local minimum in $O(n)$ time. (Note that this is faster than linear, since there are n^2 entries in the array.)