Problem Set 1

1. Let b be a fixed real number bigger than 1. Show that

$$\log_b n = \Theta(\log_2 n).$$

(This is why we often omit the bases on logs when using asymptotic notation.)

2. (Big-O practice.) For each of the following functions, describe their asymptotics as simply as possible using Θ notation. Then rank them in increasing order of growth, and indicate pairs of functions which have the same order of growth. You don't need to give proofs.

$$\begin{aligned}
a(n) &= n^2 + n3^n \\
b(n) &= \log(\log n) + 8\log n \\
c(n) &= \sqrt{n}\log^3 n + 3n\log n \\
d(n) &= n^2\log n - 112n^2 + 23 \\
e(n) &= \sqrt{n} + n\log^{100} n \\
f(n) &= \frac{n}{\log n + 1} \\
g(n) &= n\log(n^2) \\
h(n) &= n^{1/\log n} \\
i(n) &= \sqrt{7^n} - n^{10} \\
j(n) &= 10^{10} \\
k(n) &= n\log(2^n + n^3) \\
\ell(n) &= \max\left(\sqrt{n^2 + 5n}, n^{2/3} + 1000\right)
\end{aligned}$$

3. Suppose that you write out all the numbers from 1 to N in base ten. Roughly how many total digits do you have to write down? Give a Θ bound and show that your Θ bound is correct. (This means that you

should show both an upper bound and a lower bound on the number of digits. The proofs are short if you set things up nicely.)

4. The following pseudocode prints out the lyrics to the N days of Christmas, optimized for someone who really likes golden rings.

```
for d = 1, ..., N do
    Print(On the d-th day of Christmas my true love gave to me);
    for k = d, ..., 1 do
        if k = 1 then
            Print(A partridge in a pear tree);
        else
            Print(k golden rings);
        end
    end
end
```

   As a function of N give a tight big-O bound on the number of characters that this code prints. Assume that all numbers are printed in base ten. While your bound should be optimal, just give a short proof for the big-O half. (So you don't need to give a proof that your estimate is a lower bound on the number of characters printed.) This only requires a couple sentences.

5. You're given two identical eggs and an N story building. You're tasked with finding the lowest floor of the building from which the eggs will break when dropped. To do this, you're allowed to drop the eggs from a sequence of floors of your choice. Of course, once an egg breaks, you can no longer use it.

   (a) Give a strategy which only requires you to drop the eggs $O(\sqrt{N})$ times.

   (b) Show that this strategy is asymptotically optimal. That is, show that any strategy for this problem requires at least $\Omega(\sqrt{N})$ drops in the worst case.

   (c) (Optional) What if you have k eggs?

6. Read through the notebook and solve the problems/answer the questions at the end. To do your assignment you'll first make a copy of my notebook. To submit your assignment, you'll download the completed

.ipynb file (under file → download). The .ipynb file is the thing to submit.

Alternatively you can download the .ipynb file directly from the course web page.

You can work either on Colab or on your own machine. (Note that the code will probably run faster on your own machine.)